Ergodic Theory and Measured Group Theory

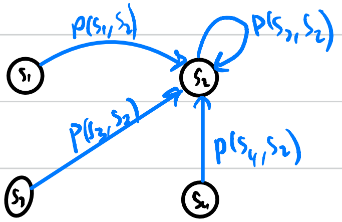
Lecture 12

An example. $S := \{a, a^{-1}, b, b^{-1}\}$ $\pi := \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

$$P := \begin{array}{c|cccc} & a & a^{-1} & b & b^{-1} \\ \hline a & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ a^{-1} & 0 & 1/3 & 1/3 & 1/3 \\ b & 1/3 & 1/3 & 1/3 & 0 \\ b^{-1} & 1/3 & 1/3 & 0 & 1/3 \end{array}$$

The induced Markov measure is supported on $\mathbb{F}_2$ and this is what was used in the Grigorchuk–Nevo thm.

Stationarity. Let S be a state space and π, P as before. We think of the Markov chain as a random walk on S, where we start with $s \in S$ with probability $= \pi(s)$ and move to s' with prob. $P(s, s')$. Thus, π is the initial distribution. What's the distribution after one step, i.e. after one step, what's the prob that we are in $s \in S$?



$$\underset{\pi_1(s)}{\underset{!!}{\mathbb{P}[w_1 = s]}} = \underset{\pi \cdot P^{(s)}}{\underset{||}{\sum_{w_0 \in S} \pi(w_0) \cdot P(w_0, s)}}.$$

Hence $\pi_1 = \pi \cdot P$.

after two steps: $\pi_2 = \pi_1 \cdot P = \pi P^2 \ldots$

Def. The Markov chain given by (π, P) is stationary if $\pi = \pi \cdot P$. Equiv., the distribution at what state the walk is doesn't change.

Markov measure on $S^{\mathbb{N}}$. A Markov chain on a state space S, denoted by m, is a prob measure on each S^n. For any basic clopen set $[w]$, where $w \in S^{<\mathbb{N}}$, define $\mathbb{P}_m[w] := m(w)$.

Then $\mathbb{P}_m[w] = \sum_{s \in S} \mathbb{P}_m[ws]$, which shows that $\mathbb{P}_m$ is a probability premeasure on the algebra of clopen sets, hence extends to a prob measure on all Borel sets by Caratheodory's theorem.

Recall that shift $s : S^{\mathbb{N}} \to S^{\mathbb{N}}$ is a Borel function and we want to understand when a Markov measure on $S^{\mathbb{N}}$ is s-invariant.

Prop. A Markov measure $\mathbb{P}_m$ on $S^{\mathbb{N}}$ is stationary $\Leftrightarrow$ it is shift-invariant.

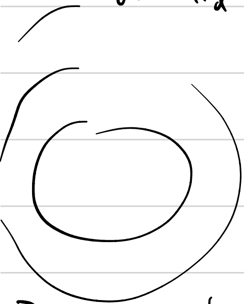
Proof. Take a basic set $[w]$ and note that $s^{-1}[w] = \bigsqcup_{t \in S} [tw]$.

It's an exercize to show that $\mathbb{P}_m(S^{-1}[w]) = \sum_{t\in S} \mathbb{P}_m([tw])$ $\iff$ $\pi P = \pi$. □

Def. A stochastic matrix P (i.e. rows add-up to 1 and nonnegative) is called irreducible if the prob. of transitioning from any $a \in S$ to any $b \in S$ in some number of steps is positive, i.e. $P^k(a,b) > 0$ for some $k \geq 1$.

Prop. A Markov measure on $S^{\mathbb{N}}$ is shift-invariant and ergodic $\iff$ $\pi P = \pi$ and P is irreducible.

Bufetov's theorem (2000). Let $\mathbb{F}_d = \langle S \rangle$, $d<\infty$, and let m be a Markov measure on $S^{\mathbb{N}}$ that is strictly irreducible and stationary. Then for every pmp action of $\mathbb{F}_d$ on (X,μ), every $f \in L^1(X,\mu)$,

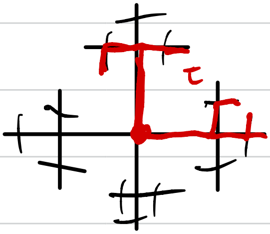


$$\lim_{n\to\infty} \frac{\sum_{\gamma\in B_n} f(\gamma\cdot x)\, m(\gamma)}{\underset{m(B_n)\,=}{n+1}} = \mathbb{E}(f : \mathcal{B}_{inv}) \text{ a.e.}$$

This generalizes the Grigorchuk-Nevo theorem. Moreover, the actual statement is for actions of free semigroup (finitely generated).

Theorem (Zomback–Ts). Let $\mathbb{F}_d := \langle S \rangle$, where $d < \infty$ and S is the standard symmetric set of generators.
Let m be a stationary Markov measure on $S^{<\mathbb{N}}$, supported on $\mathbb{F}_2$ exactly. Then for any pmp action of $\mathbb{F}_d$ on (X,μ) and any $f \in L^1(X,\mu)$,

$$\lim_{m(\tau)\to\infty} \frac{\sum f(\gamma\cdot x)\cdot m(\gamma)}{m(\tau)} \overset{a.e.}{=} \mathbb{E}(f \mid \mathcal{B}_{inv}).$$

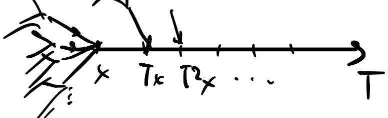


where τ ranges over $\overset{\wedge}{\text{finite}}$ subtrees of $\mathrm{Cay}(\mathbb{F}_d, S)$ containing the identity.

This is implied by our backward ergodic theorem that we will now try to state.

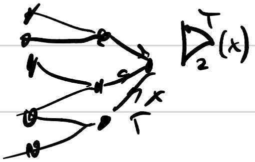
Backward ergodic theorem. Let T be a ctbl-to-one pmp transformation on (X,μ). Recall that the classical ptwise erg. theorem says that $\forall f \in L^1(X,\mu)$, for a.e. $x \in X$,

$$\lim_{n\to\infty} \text{average of } f \text{ over } I_n^T \cdot x = \mathbb{E}(f \mid \mathcal{B}_T).$$



Backward D theorem. $\lim_{n\to\infty}$ weighted average of f over $D_n^T(x) = \mathbb{E}(f|\mathcal{B}_T)$,

where $D_n^T(x) = \{x\} \cup T^{-1}x \cup \dots \cup T^{-n}x$.

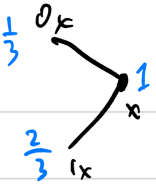


What are the weights? By the Lusin–Novikov uniformization theorem (learn DST!), there are Borel right inverses $(\gamma_n)_{n\in\mathbb{N}}$ to T s.t. $T^{-1}(x) = \{\gamma_n x : n \in \mathbb{N}\}$. E.g. in case of T being the shift on $2^{\mathbb{N}}$, $\gamma_0(x) := 0x$ & $\gamma_1(x) := 1x$. These γ_n are not measure preserving, but we may assume WLOG (learn DST!) that $\gamma_{n*}\mu \ll \mu$. Thus, there is a Radon–Nikodym derivative $\frac{d\gamma_{n*}\mu}{d\mu}(x)$ which we treat as the weight of $\gamma_n \cdot x$ relative to x. That gives a Borel function $w : E_T \to \mathbb{R}^+$ mapping $(x,y) \mapsto w_x(y)$ satisfying $w_x(y) \cdot w_y(z) = w_x(z)$. This w corrects the noninvariance of μ under the right-inverses:

$$\mu(\gamma_n(A)) = \int_A w_x(\gamma_n x)\, d\mu(x).$$

This w is called the Radon-Nikodym cocycle of E_T with respect to μ. E.g. for the shift s on $2^{\mathbb{N}}$ with $\mu := \{\overset{0}{\tfrac{1}{3}}, \overset{1}{\tfrac{2}{3}}\}$,

$w_x(0_x) = \frac{1}{3}$ & $w_x(1_x) = \frac{2}{3}$.



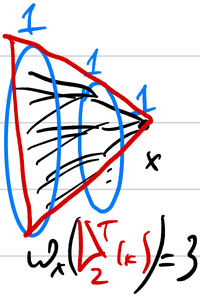
In the above theorem,

weighted-average of f over $\Delta_n^T(x)$ $:=$ $\dfrac{\sum_{y\in\Delta_n^T(x)} f(y)\cdot w_x(y)}{w_x(\Delta_n^T(x))}$.

Because T preserves μ, $\sum_{y\in T^{-1}(x)} w_x(y) = 1$.

Thus, $w_x(\Delta_n^T(x)) = n+1$. Then:

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{y\in\Delta_n^T(x)} f(y)\cdot w_x(y) = \mathbb{E}(f\mid\mathcal{B}_T).$$



Backward ergodic with trees. .... for a.e. x, let τ_x range over tree behind x (in the direction of T^{-1}) rooted at x, then:

$$\lim_{w_x(\tau_x)\to\infty} \frac{\sum_{y\in\tau_x} f(y)\, w_x(y)}{w_x(\tau_x)} = \mathbb{E}(f\mid\mathcal{B}_T).$$